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***Note on the Theory of Simultaneous Linear Differential or Difference Equations with Constant Coefficients.***

BY J. J. SYLVESTER.

This theory is virtually the same for differential as for finite-difference equations. The mere verbal part of the exposition being somewhat easier for the former of the two, I shall prefer in the first instance to deal with them, although the applications are more interesting when made to bear on the latter. Simple to the last degree as are the method of solution and the nature of the result, I do not find the one or the other set out, or even indicated, except in the most perfunctory manner, in the ordinary text-books. This brief notice, designed for the junior readers of the Journal, is intended to supply the lacuna.

Let  $u_{j,k}$  denote a linear function, with constant coefficients, of  $u_k$  and of its first  $\varepsilon_j$  derivatives in respect to  $t$ .

Let

$$\begin{array}{l} u_{1,1} + u_{1,2} + \dots + u_{1,i} = 0 \\ u_{2,1} + u_{2,2} + \dots + u_{2,i} = 0 \\ \vdots \\ u_{i,1} + u_{i,2} + \dots + u_{i,i} = 0 \end{array}$$

be the system of differential equations proposed for integration.

Call

$$\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_i = \sigma.$$

The process of arriving at the reducing equation for any one of the variables is after the manner of the dialytic method of elimination, viz.:

Along with the first equation take each of its  $(\sigma - \varepsilon_1)^{\text{th}}$  derivatives, with the second equation each of its  $(\sigma - \varepsilon_2)^{\text{th}}$  derivatives, . . . and with the  $i^{\text{th}}$  equation each of its  $(\sigma - \varepsilon_i)^{\text{th}}$  derivatives.

There will thus come into existence  $(i+1)\sigma - (\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_i)$  *i. e.*  $i(\sigma+1) - \sigma$  equations between the  $i(\sigma+1)$  quantities

$$\begin{array}{ccccccc} \omega_1, & \delta_t \omega_1, & . & . & . & . & \delta_t^\sigma \omega_1 \\ \omega_2, & \delta_t \omega_2, & . & . & . & . & \delta_t^\sigma \omega_2 \\ . & . & . & . & . & . & . \\ \omega_i, & \delta_t \omega_i, & . & . & . & . & \delta_t^\sigma \omega_i \end{array}$$

If we omit those which appear in any one of the lines above written, there will remain  $(\sigma + 1)(i - 1)$  or  $i(\sigma + 1) - \sigma - 1$  which might be eliminated between the  $i(\sigma + 1) - \sigma$  equations, and there would thus result an equation between the quantities contained in the omitted line. This elimination, it will presently be seen, there is no occasion to perform; the noticeable algebraical fact about it is, that supposing it were performed, the form of the equation resulting between  $\omega_k, \delta_t \omega_k, \dots \delta_t^\sigma \omega_k$  is invariable, whichever of the numbers 1, 2, 3, . . .  $i$  be the value assigned to  $k$ .

Let the order of the highest derivative of each  $\omega$  be reduced by one unit below the highest order previously taken, then there will be  $i\sigma - \sigma$  or  $(i - 1)\sigma$  equations connecting the  $i\sigma$  quantities

$$\begin{array}{ccccccc} \omega_1, & \delta_t \omega_1, & \dots & \delta_t^{\sigma-1} \omega_1 \\ \omega_2, & \delta_t \omega_2, & \dots & \delta_t^{\sigma-1} \omega_2 \\ \dots & \dots & \dots & \dots \\ \omega_i, & \delta_t \omega_i, & \dots & \delta_t^{\sigma-1} \omega_i \end{array}$$

and accordingly, if we omit the  $\sigma$  quantities which appear in any one (say the first) of the above lines, the remaining  $(i - 1)\sigma$  quantities may each of them be expressed as linear functions of  $\omega_1$  and its  $(\sigma - 1)$  derivatives: but the elimination previously indicated would lead to a homogeneous linear equation between  $\omega_1$  and its  $\sigma$  derivatives, and if in that, each argument  $\delta_t^\lambda \omega_1$  be replaced by  $h^\lambda$  and  $\lambda_1, \lambda_2, \dots \lambda_\sigma$  be the  $\sigma$  roots of the algebraical equation so formed, it follows from the ordinary theory for a single equation that  $\omega_1$  (provided the given equations, and consequently the resulting ones, be left in their general form) will be of the form  ${}^1C_1 e^{h_1 t} + {}^1C_2 e^{h_2 t} + \dots + {}^1C_\sigma e^{h_\sigma t}$ , and consequently by virtue of the previous remark  $\omega_2, \omega_3, \dots \omega_i$  will be of the same form as  $\omega_1$  (but, of course, with different coefficients), that is to say, the  $\sigma$  roots  $h_1, h_2, \dots h_\sigma$  are the same for the equation in  $\sigma_k$  as for the equation in  $\sigma_1$ , so that the coefficients in the equation between  $\omega_k$  and its  $\sigma$  derivatives are, as premised, independent of the value of  $k$ .

Finally, to determine the equation whose roots are  $h_1, h_2, \dots h_\sigma$ , let  ${}^1C e^{h t}$ , one of the terms in the general value, be taken as a particular value of  $\omega_1$ , which with corresponding values of the other  $\omega$ 's will serve to satisfy the given equations;  $\omega_2, \omega_3, \dots \omega_i$  being each of them linear functions of  $\omega$  and derivatives of  $\omega$ , must be of the forms  ${}^2C e^{h t}, {}^3C e^{h t}, \dots {}^iC e^{h t}$ , so that  $\omega_1, \omega_2, \dots \omega_i$  and the derivatives of each of them will contain the common factor  $e^{h t}$ , and by



which form the proposition (by virtue of Euler's method of multipliers) becomes so nearly intuitive as to abrogate all necessity for any other demonstration.\*

To pass to the parallel and more important theory in finite differences, it is only necessary to interpret  $u_{j,k}$  to signify a linear function, with constant coefficients, of  $(\omega_k)_t, (\omega_k)_{t+1}, \dots (\omega_k)_{t+\varepsilon_j}$ , where  $t$  is the integer independent variable, (say  $(\omega_k)_t$  and its  $\varepsilon_j$  difference-augmentatives), and instead of taking the differential derivatives of any one of the given equations, to take the corresponding difference-augmentatives. Then by precisely the same reasoning as before we shall have

$$\omega_{t+\sigma} + B\omega_{t+\sigma-1} + \dots + L\omega_t = 0,$$

$B, C, \dots L$  being so taken as that  $h^\sigma + Bh^{\sigma-1} + \dots + L$  shall be the determinant represented by the same form of matrix expressed by  $R$ 's as before, but where  $R_{p,q}$  is obtained from  $u_{p,q}$  by writing  $h^\theta$  in lieu of any argument  $\omega_t + \theta$  which occurs in it.

The simplest example that can be given is where  $i = 2, \varepsilon_1 = \varepsilon_2 = 1$ ,

$$\begin{aligned} u_{1,1} &= -\eta_{t+1} + a\eta_t, & u_{1,2} &= b\theta_t, \\ u_{2,1} &= c\eta_t & u_{2,2} &= -\theta_{t+1} + d\theta_t; \end{aligned}$$

this was the case which occurred in the article on the extension of Tchebycheff's theorem, in the last number of the Journal, leading to the equation

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0$$

and to expressions for  $\eta_t, \theta_t$  as linear functions of  $\lambda_1^t, \lambda_2^t$ .

It may also be remarked that this same case gives an instantaneous solution of the problem proposed and successfully treated by Babbage in his *Calculus of Functions*, more than half a century ago, and since revived in connection with the theory of substitutions (Serret, *Alg. Sup.* 4 ed. tom. 2, pp. 256–262). The problem is to find  $\phi x = \frac{ax + a}{\beta x + b}$  so that  $\phi^i x$ , say  $\frac{a_i x + a_i}{\beta_i x + b_i}$ , shall equal  $x$  for a given value of  $i$ .

To find in general  $\phi^i x$  it is only necessary to solve the difference equations

$$\begin{aligned} u_i &= \alpha u_{i-1} + \alpha v_{i-1} \\ v_i &= \beta u_{i-1} + b v_{i-1}, \end{aligned}$$

and then  $u_i, v_i$  will, if  $u_0 = 1, v_0 = 0$ , coincide with  $\alpha_i, \beta_i$ , and if  $u_0 = 0, v_0 = 1$  with  $\alpha_i, b_i$ .

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\* I regret that this simple reflection did not present itself to my mind before the preceding investigation, the necessity for which it does away with, had been set up in print. It of course applies equally well to the analogous proposition for finite-difference equations ( $u_i, v_i, \dots$  being substituted for  $x, y, \dots$ , and  $1 + \Delta$  for  $\delta_i$ ). This last named proposition, limited to the case of equations of the first order, is the foundation-stone of my new theory of Matrices regarded as Quantities, *i. e.* as subject to every kind of functional operation which ordinary arithmetical or algebraical quantities are or can be subject to: but though so important and so easily established, I know not where it can be found explicitly stated.

Thus calling  $\rho_1, \rho_2$  the two roots of

$$\begin{vmatrix} -\rho + a & \alpha \\ \beta & -\rho + b \end{vmatrix} = 0,$$

$\alpha_i$  will be of the form  $C(\rho_1^i - \rho_2^i)$  and  $\beta_i$  of the same form except as to  $C$ , say  $\Gamma(\rho_1^i - \rho_2^i)$ . Also  $a_i, b_i$  will be of the forms  $C_1\rho_1^i + C_2\rho_2^i, \Gamma_1\rho_1^i + \Gamma_2\rho_2^i$ , where  $C_1 + C_2 = 1, \Gamma_1 + \Gamma_2 = 1$ , and the required condition will be fulfilled, provided only that  $\rho_1^i = \rho_2^i$ , or say

$$\begin{aligned} \rho_1 &= K \left( \cos \frac{\lambda\pi}{i} + \sqrt{-1} \sin \frac{\lambda\pi}{i} \right) \\ \rho_2 &= K \left( \cos \frac{\lambda\pi}{i} - \sqrt{-1} \sin \frac{\lambda\pi}{i} \right) \end{aligned}$$

*i. e.* if  $(a + b)^2 - 4(ab - \alpha\beta) \left( \cos \frac{\lambda\pi}{i} \right)^2 = 1$ ,  $\lambda$  having any integer value (which without loss of generality may be taken inferior to  $i$ ) except zero.\*

If  $\lambda = 0$ , the two roots of the equation in  $\rho$  become equal and the form of the solution changes into

$$u_i = (C_1 + C_2 i) \rho^i \quad v_i = (C'_1 + C'_2 i) \rho^i.$$

When  $u_0 = 1$  and  $v_0 = 0$  then  $u_1 = a, v_1 = \beta$ ,

$$C_1 = 1, C'_1 = 0, \quad C_2 = \frac{a}{\rho} - 1, C'_2 = \frac{\beta}{\rho},$$

and when  $u_0 = 0, v_0 = 1, u_1 = \alpha, v_1 = b$ ,

$$C_1 = \frac{\alpha}{\rho}, C'_1 = \frac{b}{\rho} - 1, \quad C_2 = 0, C'_2 = 1,$$

and  $\phi^i x = \frac{[\rho + i(a - \rho)]x + i\alpha}{i\beta x + \rho + (b - \rho)i}$  which cannot be periodic for any value of  $i$ ,

and when  $i = \infty$  becomes  $\frac{(a - \rho)x + \alpha}{\beta x + b - \rho} = \frac{a - \rho}{\beta} = \frac{a}{b - \rho}$ , *i. e.*  $= \frac{a - b}{2\beta}$  or  $\frac{2a}{a - b}$ , so that  $\phi^i x$  in this case continually converges to a constant limit.

I may add that  $\phi^i x$  converges to a constant limit not merely when the roots  $\rho_1, \rho_2$  of

$$\begin{vmatrix} a - \rho & \alpha \\ \beta & b - \rho \end{vmatrix}$$

are equal, but whenever they are real. For the general form of  $\phi^i x$ , it may easily be found, is

$$\frac{[(\rho_2 - a)\rho_1^i + (\rho_1 - a)\rho_2^i]x + a(\rho_1^i - \rho_2^i)}{\beta(\rho_1^i - \rho_2^i)x + [(\rho_2 - b)\rho_1^i + (\rho_1 - b)\rho_2^i]}$$

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\* There will thus be  $(i - 1)$  values of  $\lambda$  which will each give a distinct admissible solution of the problem of periodicity, but of course only those values of  $\lambda$  which are relatively prime to  $i$  will give primitive solutions. If  $i = i'\delta$  the effect of making  $\lambda = \lambda'\delta$  will be to make  $\phi^i x = x$  by virtue of its making  $\phi^{i'} x = 0$ .

which if  $\rho_2 > \rho_1$  when  $i = \infty$  becomes  $\frac{(\rho_1 - a)x - a}{-\beta x + \rho_1 - b} = \frac{a - \rho_1}{\beta}$  or  $\frac{a}{b - \rho_1}$  where  $\rho_1$  signifies the smaller of the two roots  $\rho_1, \rho_2$ ; or in other words when  $a - b > 2\sqrt{a\beta}$ , the limiting value to  $\phi^i x$ , when  $\phi x$  represents  $\frac{ax + a}{\beta x + b}$ , is  $\frac{(a-b) + \sqrt{(a-b)^2 - 4a\beta}}{2\beta}$ , with the understanding that the quantity under the radical sign is to be taken positive.

So, if

$x_{i+1} : y_{i+1} : z_{i+1} = ax_i + by_i + cz_i : a'x_i + b'y_i + c'z_i : a''x_i + b''y_i + c''z_i$ ,  
when all the roots of the determinant

$$\begin{vmatrix} a - \lambda & b & c \\ a' & b' - \lambda & c' \\ a'' & b'' & c'' - \lambda \end{vmatrix}$$

are real, the point  $x_i, y_i, z_i$ , as  $i$  increases, will be found to approach indefinitely near to a fixed straight line; and if all the roots are equal, to a fixed point.

The condition of the system of ratios  $x_i : y_i : z_i$  being periodic and having a period  $m$  is tantamount to the condition that the  $m^{\text{th}}$  power of the matrix

$$\begin{matrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{matrix}$$

shall be the matrix

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1. \end{matrix}$$

The complete solution of this problem, and of the more general one of extracting the  $m^{\text{th}}$  root of any unit-matrix (*i. e.* a matrix in which each element in the principal diagonal is unity, and the rest zero), which constitutes the ultimate generalization of Babbage's problem and is soluble by the same method, will probably appear in a memoir on matrices, in the forthcoming number of the Journal.

In general, for a matrix of the order  $\omega$ , the number of  $m^{\text{th}}$  roots is  $m^\omega$  and each of them is perfectly determinate. But when the matrix is a unit-matrix or a zero-matrix (the latter meaning one in which every element is zero) there are distinct genera and species of such roots, and every species contains its own appropriate number of arbitrary constants.